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Holomorphic Differentials as Functions of Moduli

LIPMAN BERS

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The purpose of this note is to strengthen the results of [3] and to indicate a very brief derivation of some theorems announced without proof in [1] and [3].

We begin by indicating a correction to [3]. Let S_1 and S_2 be Riemann surfaces, f an orientation preserving (orientation reversing) homeomorphism of bounded eccentricity of S_1 onto S_2 and $[f]$ the homotopy class of f ; then $(S_1, [f], S_2)$ is called an even (odd) coupled pair of Riemann surfaces. The definition of equivalence of such pairs given in [3] is imprecise and garbled by misprints. The correct definition reads: $(S_1, [f], S_2)$ and $(S'_1, [f'], S'_2)$ are called equivalent if there exist conformal homeomorphisms h_1 and h_2 with $h_1(S_1) = S'_1, h_2(S_2) = S'_2$ and $[h_2 f] = [f' h_1]$; the two pairs are called strongly equivalent if $S'_2 = S_2$ and there exists a conformal homeomorphism h with $h(S_1) = S'_1$ and $[f] = [h' f]$. If S_0 is a Riemann surface, then the Teichmüller space $T(S_0)$ can be thought of as the set of strong equivalence classes of even pairs $(S, [f], S_0)$ (and not of simple equivalence classes as stated in [3]).¹

From now on we assume that S_0 is a fixed closed Riemann surface of genus $g > 1$, and we write T instead of $T(S_0)$. T has a natural complex analytic structure and can be represented as a bounded domain, homeomorphic to a ball, in complex number space with coordinates (moduli) $\tau_1, \dots, \tau_{3g-3}$ (cf. [1, 2]). Points of T will be denoted by τ . We may assume that S_0 is given as the unit disc modulo a fixed-point-free Fuchsian group, and that $\tau=0$ corresponds to the pair $(S_0, [\text{identity}], S_0)$.

THEOREM I. One can associate to every $\tau \in T$ a bounded Jordan domain $D(\tau)$ and $2g$ Möbius transformations $z \rightarrow A_j(z, \tau)$, $z \rightarrow B_j(z, \tau)$, $j = 1, \dots, g$, such that the following conditions are satisfied.

(i) The boundary curve of $D(\tau)$ admits the parametric representation $z = \sigma(e^{i\theta}, \tau)$, $0 \leq \theta \leq 2\pi$, depending holomorphically on τ . $D(0)$ is the unit disc.

(ii) The A_j and B_j depend holomorphically on τ and satisfy the relation

$$(1) \quad \prod_{j=1}^g A_j B_j^{-1} = 1$$

For every fixed $\tau \in T$ they generate, with the single defining relation (1), a fixed-point-free discrete group $G(\tau)$ of conformal self-mappings of $D(\tau)$, so that $S(\tau) = D(\tau) / G(\tau)$ is a closed Riemann surface of genus g . $S(0)$ is the surface S_0 .

(iii) Denote by $\alpha(\tau)$ the basis of the fundamental group of $S(\tau)$ defined by A_1, \dots, B_g , and by f_τ a quasiconformal mapping of $S(\tau)$ onto $S(0)$ which takes $\alpha(\tau)$ into $\alpha(0)$. Then the point τ corresponds to the pair $(S(\tau), [f_\tau], S_0)$.

This statement differs from Theorem 2 in [3] primarily by the boundedness condition for $D(\tau)$ and can be obtained from that theorem without much difficulty.

We denote by M the domain in complex number space of $3g-2$ dimensions which consists of points (z, τ) with $z \in D(\tau)$ and $\tau \in T$. By Theorem 3 in [3] M is holomorphically equivalent to $T(S_0 - \{p\})$ for a fixed $p \in S_0$.

and the first of the following year. The first
of the year 1872, however, he had to leave
the country, and was compelled to go to
the United States, where he remained
for about two years. In 1874 he
returned to England, and in 1876
settled in New York City, where he
lived until his death. He died at
his residence in New York on the
21st of January, 1881, at the age of
seventy-four years. His remains were
interred in the cemetery of the
Episcopal Church of St. John the
Divine, in New York City. He
leaves a widow, Mrs. Mary E.
Hawkins, and a son, Dr. Wm. H.
Hawkins, of New York City.

We denote by $W_q(\tau)$ the (complex) vector space of holomorphic functions $\varphi(z)$, $z \in D(\tau)$, for which $\varphi(z)dz^q$ is invariant under $G(\tau)$; this is the same as the space of q -dimensional holomorphic differentials on $S(\tau)$, so that $\dim W_q(\tau) = 0, g$, or $(2q-1)(g-1)$ according to whether $g \leq 0$, $g=0, g \geq 1$, or $g > 1$. In $W_1(\tau)$ there exist g distinguished elements, $p_k(z, \tau)$, determined by the conditions

$$(2) \quad A_i(z, \tau) = \int_z p_k(z', \tau) dz' = \int_{ik};$$

these correspond to the normalized Abelian differentials of the first kind on $S(\tau)$ belonging to the 'canonical' homology basis $a(\tau)$ determined by $\alpha(\tau)$. The period matrix of $S(\tau)$ belonging to $a(\tau)$ will be denoted by $Z(\tau)$. It has the elements

$$Z_{ik}(\tau) = \int_z B_i(z, \tau) dz'$$

and is a point in the Siegel space of symmetric matrices with positive definite imaginary part.

We denote by W_{-q} the vector space of holomorphic functions $\tilde{\varphi}(z, \tau)$, $(z, \tau) \in M$, which belong to $W_q(\tau)$ for every fixed $\tau \in T$.

THEOREM II. Every element of $W_q(\tau)$ is a restriction of an element of W_{-q} .

Proof. Assume that $q > 2$. Let C_j , $j=1, 2, \dots$, be a complete system of non equivalent (with respect to (1)) words in the letters A_1, \dots, B_g . If $P(t)$ is a polynomial, then the Poincaré series

$$(3) \quad \sum_{j=1}^{\infty} P(C_j(z, \tau)) (\partial C_j(z, \tau) / \partial z)^q$$

the first time in the history of the world, the
whole of the human race has been gathered
together in one place, and that is the
present meeting of the World's Fair.
The great number of people here
from all parts of the world,
and the great variety of
things exhibited,
make it a most interesting
and instructive meeting.
The exhibits are
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converges normally in M and its sum belongs to \underline{W}_q . On the other hand, since $D(\tau)$ is a bounded Jordan domain and $G(\tau)$ has a compact fundamental region, Theorem 4 in [4] implies that, for a fixed τ , every element of $\underline{W}_q(\tau)$ is of the form (3).

For $q = 1$ we shall show that every p_j belongs to \underline{W}_1 (i.e. that the normalized Abelian differentials are holomorphic functions of the moduli).

THEOREM III. The functions $p_k(z, \tau)$, $k=1, \dots, g$, are holomorphic in M .

Proof. It suffices to consider p_1 . We shall show that in a neighborhood of a fixed but arbitrary point $\tau_0 \in T$ we have an identity of the form

$$(4) \quad p_1(z, \tau) = \bar{\phi}(z, \tau)^{-1} \sum_{j=1}^{5g-5} c_j(\tau) \bar{\phi}_j(z, \tau)$$

where the c_j are holomorphic, $\bar{\phi} \in \underline{W}_2$, and the $\bar{\phi}_j$ are elements of \underline{W}_3 . We first choose $\bar{\phi}$ so that $\bar{\phi}(z, \tau_0)$ vanishes at $4g-4$ points z_i which are not equivalent under $G(\tau_0)$. This is possible since the 'general' holomorphic quadratic differential on $S(\tau)$ has only simple zeros (Bertini) and hence exactly $4g-4$ of those. There exist $4g-4$ holomorphic functions $\beta_i(\tau)$ defined near τ_0 , such that $\beta_i(\tau_0)=0$ and $\bar{\phi}(z_i + \beta_i(\tau), \tau) = 0$. In order that the right hand side of (4) belong to $\underline{W}_1(\tau)$ it is necessary and sufficient that

$$\sum_{j=1}^{5g-5} c_j(\tau) \bar{\phi}_j(z_i + \beta_i(\tau), \tau) = 0, \quad i = 1, \dots, 4g-4,$$

and one sees at once that any $4g-5$ of these equations imply the $(4g-4)$ -th. In order that (4) hold near τ_0 the c_j must satisfy g

THE INFLUENCE OF CULTURE ON LANGUAGE

and the language of the people. In this way, the language of the people becomes the language of the country.

It is also important to note that the language of the people is the language of the country.

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additional linear equations which are obtained from (1) by setting $k=1$ and choosing a fixed point z and fixed paths of integration, avoiding the points z_1 . The resulting linear system, with holomorphic coefficients, for the unknown functions c_j , is uniquely solvable at τ_0 if the functions $\tilde{\phi}_1, \dots, \tilde{\phi}_{15g-5}$ are chosen so as to be linearly independent for $\tau = \tau_0$. In this case the equations are uniquely solvable for τ close to τ_0 , and the solutions depend holomorphically on τ .

We proceed to derive some consequences from Theorems II and III.

(a) The functions

$$f_{ij} = p_i/p_j, \quad f_{ijk} = f_k^{-1} \partial \log f_{ij} / \partial z$$

are meromorphic in M . This proves Theorem J in [1]. It is classical that every meromorphic function in $D(\tau)$ which is automorphic under $G(\tau)$ can be expressed rationally in terms of the functions f_{ij}, f_{ijk} (and even in terms of the f_{ij} alone if $S(\tau)$ is not hyperelliptic). Thus we obtain a proof of Theorem 4 in [3] which asserts the existence of finitely many meromorphic functions of the moduli and of an additional complex variable which uniformize simultaneously all algebraic curves of genus $g > 1$.

(b) Let us choose $(2q-1)(g-1)$ elements of $\underline{W}_q, q > 1$ (or g elements of \underline{W}_1) which are linearly independent for $\tau = \tau_0$, and let $w(z, \tau)$ denote their Wronskian with respect to z . For a fixed τ close to τ_0 the zeros of $w(z, \tau)$ are precisely the Weierstrass points of $S(\tau)$, in the classical sense if $q = 1$, in the sense of Petersson if $q > 1$ (cf. the definition in [4]). Since w is a holomorphic function in M

and the *lungs* were *normal*. The *liver* was *normal*.

The *urine* was *normal*. The *stool* was *normal*.

The *skin* was *normal*. The *hair* was *normal*.

The *teeth* were *normal*. The *gums* were *normal*.

The *eyes* were *normal*. The *ears* were *normal*.

The *hands* were *normal*. The *feet* were *normal*.

The *abdomen* was *normal*. The *rectum* was *normal*.

The *bladder* was *normal*. The *kidneys* were *normal*.

we conclude that the Weierstrass points on a closed Riemann surface depend holomorphically on the moduli (cf. Rauch [6], Röhrl [7]).

(c) Now let $w(z, \tau)$ denote the Wronskian of an arbitrary set of $\dim W_q(\tau)$ elements of W_q and let N denote the set of those $\tau \in T$ for which $w(z, \tau) \equiv 0$. If z_0 is not a Weierstrass point of $S(\tau_0)$, then there is a neighborhood of τ_0 in which the points of N are precisely the zeros of $w(z_0, \tau)$. We conclude that N is either empty, or the whole domain T , or an analytic subvariety of T of codimension 1.

(d) Let H denote the set of those $\tau \in T$ for which $S(\tau)$ is hyperelliptic. For $\tau \in T - H$ every element of $W_q(\tau)$ can be written as a homogeneous polynomial in the p_j (M. Noether). For $\tau \in H$ the subspace of $W_q(\tau)$ consisting of homogeneous polynomials in elements of $W_1(\tau)$ has dimension $q(g-1)-1$. But H is an analytic subvariety of T of dimension $2g-1$, so that, noting (c), we obtain the following complement to Noether's theorem: for $g > 2$ and $q > 1$ there exist no fixed set of $(2q-1)(g-1)$ homogeneous polynomials of degree q in normalized Abelian differentials of the first kind which spans the space of holomorphic differentials of dimension q on all non-hyperelliptic closed Riemann surfaces of genus g .

(e) The mapping $\tau \rightarrow Z(\tau)$ of the Teichmüller space into the Siegel space is holomorphic. This follows at once from Theorem III, and also by using the coordinates in T defined in [1] in conjunction with Rauch's variational formulas [5]. These formulas also show

that the mapping of T into a $(3g-3)$ -dimensional subspace of the Siegel space,

$$\tau \rightarrow \left\{ \sum_{i,k=1}^{3g-3} \gamma_{j,ik} z_{ik}(\tau), \quad j = 1, \dots, 3g-3 \right\}$$

is one-to-one near a point τ_0 if and only if the $3g-3$ functions

$$\sum_{i,k=1}^{3g-3} \gamma_{j,ik} p_i(z, \tau_0) p_k(z, \tau_0)$$

are linearly independent. This shows that near every non-hyperelliptic surface a properly chosen set of $3g-3$ periods z_{ik} can serve as a set of local moduli (Rauch). On the other hand, (d) implies a complement to Rauch's theorem: no fixed set of $3g-3$ linear combinations of periods can serve as a set of moduli near every non-hyperelliptic closed Riemann surface of genus $g > 2$.

and the number of individuals of each species found
in each sample was determined.

RESULTS AND DISCUSSION

1. Mean Number of Individuals per Sample

The mean number of individuals per sample was calculated for each species and the results are shown in Table I. The mean number of individuals per sample ranged from 1.0 to 1.5.

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FOOTNOTE

1) We also note the following errata to [2,3]. On p. 94, 1.19, replace (\mathcal{J}) by $\mu(\mathcal{J})$. On p. 96, 1.15, replace the subscript j by $2j$. On p. 97, 1.21, replace C_r by C^r . On p. 100, 1.4, replace 'covering' by 'covering space'. On p. 103, equation (9) replace the exponent $3g-3n+n$ by $3g-3+n$.

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the same day, and was then sent to the U.S. Fish Commission, which
arrived at San Francisco on the 23rd. It was examined by Dr. G.
Brown Goode, who found it to be a new species, and gave it the name
of *Scyliorhinus stellaris*. This was published in the "Proceedings
of the U.S. National Museum," Vol. 12, No. 63, p. 253, in 1870.
The specimen was sent to the British Museum, where it is now
preserved. It is a small shark, about 3 feet long, and has a
large head, with a very long snout, and a large mouth, with
very sharp teeth. The body is elongated, and the fins are
moderately large. The color is dark brown, with a lighter
area on the side of the head, and a white area on the
ventral surface. The dorsal fin is located near the middle
of the body, and the pectoral fins are located near the
head. The caudal fin is deeply forked, and the ventral
fins are located near the base of the tail. The eyes are
large, and the nostrils are located on the sides of the
snout. The mouth is large, and the teeth are sharp.
The specimen was collected from the Pacific Ocean, off
the coast of California, and was probably taken by
a fisherman. It is a rare species, and is only known
from a few specimens.

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